

Zero-energy states of $\mathcal{N} = 4$ SYM on T^3 : S -duality and the mapping class group

Måns Henningson and Niclas Wyllard

*Department of Fundamental Physics, Chalmers University of Technology,
S-412 96 Göteborg, Sweden*

E-mail: mans@fy.chalmers.se, wyllard@fy.chalmers.se

ABSTRACT: We continue our studies of the low-energy spectrum of $\mathcal{N} = 4$ super-Yang-Mills theory on a spatial three-torus. In two previous papers, we computed the spectrum of normalizable zero-energy states for all choices of gauge group and all values of the electric and magnetic 't Hooft fluxes, and checked its invariance under the $SL_2(\mathbb{Z})$ S -duality group. In this paper, we refine the analysis by also decomposing the space of bound states into irreducible unitary representations of the $SL_3(\mathbb{Z})$ mapping class group of the three-torus. We perform a detailed study of the S -dual pairs of theories with gauge groups $Spin(2n+1)$ and $Sp(2n)$. The predictions of S -duality (which commutes with the mapping class group) are fulfilled as expected, but the proof requires some surprisingly intricate combinatorial infinite product identities.

KEYWORDS: Duality in Gauge Field Theories, Discrete and Finite Symmetries, Extended Supersymmetry, Supersymmetric gauge theory.

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1. Introduction

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory has no mass gap; the spectrum of the theory extends continuously down to zero energy. This is true not only in Minkowski space, but also if the theory is considered on $\mathbb{R} \times T^3$, where the first factor denotes time and the second is a spatial three-torus. In the weak coupling limit, the wave functions of the low-energy states are then supported on the moduli space of flat connections on the gauge bundle over T^3 . (This means that the magnetic field strength is zero.) Generically, these low-energy states break the gauge group G of the theory to an abelian subgroup, but on certain subspaces of the moduli space of flat connections, the unbroken subgroup Z may be of the form

$$Z \simeq S \times U(1)^r, \tag{1.1}$$

where S is a semi-simple group and r is some non-negative integer. Because of the scalar fields in the $\mathcal{N} = 4$ multiplets associated with the abelian $U(1)^r$ factor, the quantum states will in general not be normalizable; we refer to this as a rank r continuum of states. The effective low-energy theory associated with the semi-simple factor S is modeled by supersymmetric quantum mechanics with 16 supercharges based on the Lie algebra \mathfrak{s} of the group S [1]. The latter theory is the dimensional reduction to 0 + 1 dimensions of $\mathcal{N} = 4$ Yang-Mills theory, and is believed to have a linear space V_s of normalizable zero-energy states [2]. In the Yang-Mills theory, there are thus $\sum_s \dim V_s$ continua of rank r of low-energy states. These can be further characterized by their discrete abelian 't Hooft

fluxes [3]: The magnetic 't Hooft flux m measures the topological class of the gauge bundle, and the electric 't Hooft flux e determines (together with the θ -angle) the transformation properties of the quantum states under large gauge transformations.

In two previous papers [4, 5], we constructed the low-energy spectrum of normalizable states for all choices of a simple gauge group G (assuming the spectrum is independent of the coupling constant). In particular, we showed that the requirement of $SL_2(\mathbb{Z})$ S -duality [6] (under which the 't Hooft fluxes (m, e) transform as a doublet) determines the dimensions of the spaces V_s for all semi-simple Lie algebras s (almost) uniquely: $\dim V_s$ equals the number of distinguished markings of the corresponding Dynkin diagram. (This is in agreement with the result obtained by considering a mass-deformed version of the $\mathcal{N} = 4$ quantum mechanics and assuming that the states are independent of the mass perturbation [7].)

In this paper, we refine the analysis of the spectrum by also examining the behavior of the states under the $SL_3(\mathbb{Z})$ mapping class group of the spatial T^3 : For given values of the 't Hooft fluxes m and e , the corresponding space of quantum states may be decomposed as a direct sum of irreducible unitary representations R of the stability subgroup of the mapping class group that leaves m and e invariant. The spectrum of degeneracies of such representations should be invariant under S -duality acting on m and e , but this is not at all manifest in the present formulation of the theory. We believe that it should eventually be possible to give a simpler proof of this invariance under S -duality, valid for continua of states of arbitrary rank r and all gauge groups G . Such a result is likely to give additional insight into the structure of the theory. Here, we proceed in a more pedestrian way, though, and limit ourselves to truly normalizable zero-energy states, i.e. the $r = 0$ case, and the S -dual pair $G = \text{Spin}(2n+1)$ and $G = \text{Sp}(2n)$. (The case $G = \text{SU}(n)$ is rather trivial [4]: For given values of m and e , there is at most one state, and this transforms trivially under the stability subgroup of the mapping class group. We also have some partial results for the $G = \text{Spin}(2n)$ cases, but as they shed no particular further light on the underlying structure, we have chosen not to present them here. The cases when G is an exceptional Lie group appear to be technically complicated, but should otherwise pose no particular problems. However, it would probably be more worthwhile to try to understand the general structures, rather than proceeding in a case-by-case manner.)

After a short description of the general aspects of the theory in section two, we compute the spectrum of bound states of the $\text{Spin}(2n+1)$ and $\text{Sp}(2n)$ theories in sections three and four respectively. The results are then compared in section five, and found to agree with the predictions of S -duality. The proof reveals surprising connections to subtle combinatorial identities related to infinite product expressions for theta functions. The deeper meaning of this is still unclear to us.

2. General considerations

Let G be the gauge group with center subgroup C . By choosing a specific basis of three primitive one-cycles on T^3 , the isomorphism class of a principal G/C bundle over T^3 (the

gauge bundle) may be specified by a triple

$$m = (m_{23}, m_{31}, m_{12}) \in C^3, \tag{2.1}$$

where $m_{ij} = m_{ji}^{-1} \in C$ is identified with the restriction of the discrete abelian magnetic 't Hooft flux to a two-torus in the ij -plane. It is sometimes convenient to use the dual notation

$$m = (m_1, m_2, m_3) \in C^3. \tag{2.2}$$

A flat connection on such a bundle is determined by its holonomies along the non-trivial cycles of the torus, i.e. by a triple

$$U = (U_1, U_2, U_3) \in G^3, \tag{2.3}$$

subject to the almost commutation relations

$$U_i U_j U_i^{-1} U_j^{-1} = m_{ij}. \tag{2.4}$$

Let $[U]$ denote the equivalence class of U modulo simultaneous (gauge) conjugation of the U_i by some element of G . For a detailed description of the structure of the moduli space of flat connections, see [8].

For an almost commuting triple U , we let $Z \subset G$ denote its centralizer (the unbroken gauge group), i.e. the subgroup of elements that commute with the U_i . For simplicity, we will in this paper only be concerned with U such that $Z = S$ is semi-simple, i.e. we will not consider continua of non-zero rank r . For such a U , we let Δ be the finite set of distinguished markings of the Dynkin diagram associated to the Lie algebra s of S . As described in the introduction, there is a linear space V_s of normalizable zero-energy states in the supersymmetric quantum mechanics with 16 supercharges based on s . V_s has an orthonormal basis in one-to-one correspondence with the elements of Δ . For a fixed isomorphism class of semi-simple centralizer Z , we then get a linear space V_Z of normalizable zero-energy states with an orthonormal basis of elements denoted $[[U], \delta]$. Here $[U]$ is a conjugacy class of an almost commuting triple with semi-simple centralizer isomorphic to Z , and $\delta \in \Delta$. The total space V of bound states is the direct sum of the spaces V_Z , where the sum runs over all possible semi-simple centralizers $Z \subset G$.

The C^3 group of G/C gauge transformations with a non-trivial winding around the cycles of T^3 is a module of the $SL_3(\mathbb{Z})$ mapping class group of T^3 , so we may form the semi-direct product

$$\hat{\Omega} = SL_3(\mathbb{Z}) \ltimes C^3. \tag{2.5}$$

This group (almost) acts by permutations on the set of U , and this action preserves the centralizer Z : The action of the first factor (the mapping class group) is induced from the action on the homology of T^3 , so that the group element

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in SL_3(\mathbb{Z}) \tag{2.6}$$

acts according to

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \mapsto \begin{pmatrix} U_1^{a_{11}} U_2^{a_{12}} U_3^{a_{13}} \\ U_1^{a_{21}} U_2^{a_{22}} U_3^{a_{23}} \\ U_1^{a_{31}} U_2^{a_{32}} U_3^{a_{33}} \end{pmatrix}. \tag{2.7}$$

(This is well-defined if U is a commuting triple, but for an almost commuting triple we need to consider U modulo conjugation by elements of the finite group generated by the U_i . Such conjugations act trivially on the centralizer Z .) This implies that the components of m transform as a triplet under $\text{SL}_3(\mathbb{Z})$. The action of the second factor in (2.5) is

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \mapsto \begin{pmatrix} c_1 U_1 \\ c_2 U_2 \\ c_3 U_3 \end{pmatrix}, \tag{2.8}$$

where $c_i \in C$. Both factors act trivially on δ . This action on U and δ descends to an action on pairs $([U], \delta)$, which however may be non-trivial also on δ . Indeed, a choice of preferred representatives U for the classes $[U]$ is in general not preserved by the action of $\hat{\Omega}$, so each such transformation must be accompanied by a suitable conjugation in G . This defines an automorphism of Z , which, modulo conjugation in Z , determines a Dynkin diagram automorphism acting on δ .

The action of $\hat{\Omega}$ on the pairs $([U], \delta)$ induces a linear action on the vector space V of normalizable zero-energy states. We begin the analysis of this action by considering the factor C^3 in $\hat{\Omega}$. An irreducible linear representation of C^3 is determined by a triple (the discrete abelian electric 't Hooft flux)

$$e = (e_1, e_2, e_3) \in \tilde{C}^3 \simeq C^3, \tag{2.9}$$

transforming in the same way as m under $\text{SL}_3(\mathbb{Z})$. Here we have used the (in this context canonical) isomorphism between the finite abelian group C and its dual $\tilde{C} = \text{Hom}(C, U(1))$. We may thus decompose the space V of normalizable zero-energy states as a direct sum of subspaces, each of which is characterized by an orbit of $\text{SL}_3(\mathbb{Z})$ on the set of ordered pairs $(m, e) \in C^3 \times C^3$, together with an irreducible representation R of the 'little' subgroup $\Omega_{m,e} \subset \text{SL}_3(\mathbb{Z})$ stabilizing some chosen pair on that orbit. We let $N_{m,e}^R(G)$ denote the multiplicity of such representations. The $\text{SL}_2(\mathbb{Z})$ S -duality group is expected to commute with the $\text{SL}_3(\mathbb{Z})$ mapping class group and transforms the pair (m, e) as a doublet. Invariance of the spectrum under S -duality thus amounts to the conditions

$$\begin{aligned} N_{c,c'}^R(G) &= N_{c',c^{-1}}^R(G') \\ N_{c,c'}^R(G) &= N_{c,cc'}^R(G), \end{aligned} \tag{2.10}$$

where G' denotes the Langlands or GNO dual group of G/C . It is not obvious that the spectrum fulfills these conditions, but in the following sections we will check this explicitly for the S -dual pairs of theories with gauge groups $\text{Spin}(2n+1)$ and $\text{Sp}(2n)$.

It should be noted that $N_{m,e}^R(G)$ necessarily vanishes for certain combinations of m and e : For a given $m \in C^3$, multiplication of a triple $U = (U_1, U_2, U_3)$ by (m_{1i}, m_{2i}, m_{3i}) for some $i = 1, 2, 3$ is equivalent to conjugation by U_i , and thus acts trivially on the space

of states. The possible values of $e \in \tilde{C}^3$ are thus those that obey $e_1(m_{1i})e_2(m_{2i})e_3(m_{3i}) = 1 \in U(1)$. (This notation means that the components of e are evaluated on the components of m .) Evaluating this equation for $i = 1, 2, 3$, and using the dual notation for m , gives three equations that can be summarized as

$$e_i m_j = e_j m_i \tag{2.11}$$

for all $i, j = 1, 2, 3$. The multiplicity $N_{m,e}^R(G)$ can thus be non-zero only for m and e fulfilling this S -duality covariant constraint.

It should also be noted that, although the infinite discrete group $\hat{\Omega}$ has infinitely many inequivalent representations, only finitely many of these will appear. The reason is that the holonomies U_i with semi-simple centralizers are of finite order in G (i.e. a finite power of U_i equals the identity element). There is therefore a normal subgroup $\hat{\Omega}_0$ of finite index in $\hat{\Omega}$ that acts trivially on the holonomies, so all representations of $\hat{\Omega}$ that appear are pullbacks of representations of the finite quotient group

$$\Gamma = \hat{\Omega}/\hat{\Omega}_0. \tag{2.12}$$

Representation theory of finite groups will therefore play a central role in our analysis; for some background material on such groups see e.g. [9]. To decompose the space V of normalizable zero-energy states as a direct sum of irreducible representations of Γ , we can proceed as follows: For $\gamma \in \Gamma$ we let $[\gamma]$ denote its conjugacy class, i.e. the set of elements obtained from γ by conjugation by elements of the group. The cardinality of $[\gamma]$ is denoted $d_{[\gamma]}$, so that

$$d = \sum_{[\gamma]} d_{[\gamma]}, \tag{2.13}$$

where the sum runs over all conjugacy classes, equals the order of Γ . For each irreducible unitary representation R of Γ and each conjugacy class $[\gamma]$, the character of R evaluated on $[\gamma]$ is given by $\text{Tr}_R(\gamma)$, where γ is a representative of $[\gamma]$. The multiplicity N^R of the representation R in the decomposition of V now follows from the orthogonality properties of the characters, and is given by the formula

$$N^R = \frac{1}{d} \sum_{[\gamma]} d_{[\gamma]} \text{Tr}_V(\gamma) \text{Tr}_R(\gamma), \tag{2.14}$$

where the sum again runs over all conjugacy classes. It follows from the structure of V as described above, that the trace $\text{Tr}_V(\gamma)$ equals the number of pairs $([U], \delta)$ fixed by the action of γ . This means that $[U]$ should be invariant under γ , and that δ should be invariant under the Dynkin diagram automorphism induced by γ as described above.

3. The $G = \text{Spin}(2n + 1)$ theories

In this section we discuss the $G = \text{Spin}(2n+1)$ theories; see [4] for relevant background material. As discussed in the introduction, the bound states arise at points in the moduli

space of flat connections where the unbroken gauge group is semi-simple. Possible such semi-simple centralizers are of the form [10, 11, 4]

$$z \simeq \mathfrak{so}(k_0) \oplus \mathfrak{so}(k_1) \oplus \dots \oplus \mathfrak{so}(k_7), \tag{3.1}$$

where $k_0 + k_1 + \dots + k_7 = 2n + 1$. One may think of k_1, \dots, k_7 as being associated with the points of the Fano plane, i.e. the non-zero points of \mathbb{Z}_2^3 (which can be viewed as the corners of a cube). When $m \in C^3 \simeq \mathbb{Z}_2^3$ is trivial, there are two possibilities [10]: Either k_0 is even and the k_a for $a = 1, \dots, 7$ are odd, or vice versa. When $m \in C^3 \simeq \mathbb{Z}_2^3$ takes one of the seven non-trivial values, it determines one of the seven lines of the Fano plane. Again there are two possibilities: Either the k_a associated to the three points of that line are even whereas k_0 and the remaining four k_a are odd, or vice versa. In all cases, the images of the corresponding commuting triples in $G/C \simeq \text{SO}(2n + 1)$ may be represented by the diagonal matrices

$$\begin{aligned} \bar{U}_1 &= \text{diag}(\mathbb{1}_{k_0}, -\mathbb{1}_{k_1}, \mathbb{1}_{k_2}, -\mathbb{1}_{k_3}, \mathbb{1}_{k_4}, -\mathbb{1}_{k_5}, \mathbb{1}_{k_6}, -\mathbb{1}_{k_7}) \\ \bar{U}_2 &= \text{diag}(\mathbb{1}_{k_0}, \mathbb{1}_{k_1}, -\mathbb{1}_{k_2}, -\mathbb{1}_{k_3}, \mathbb{1}_{k_4}, \mathbb{1}_{k_5}, -\mathbb{1}_{k_6}, -\mathbb{1}_{k_7}) \\ \bar{U}_3 &= \text{diag}(\mathbb{1}_{k_0}, \mathbb{1}_{k_1}, \mathbb{1}_{k_2}, \mathbb{1}_{k_3}, -\mathbb{1}_{k_4}, -\mathbb{1}_{k_5}, -\mathbb{1}_{k_6}, -\mathbb{1}_{k_7}) \end{aligned} \tag{3.2}$$

The order of the above holonomies \bar{U}_i in G/C is 1 or 2 so that $U_i^2 \in C$. In a sector with a fixed value of e , it is therefore sufficient to consider the action on the holonomies of a finite group Γ defined as the reduction modulo 2 of the stability group $\Omega_{m,e} \subset \text{SL}_3(\mathbb{Z})$.

There are $2^3 = 8$ different liftings of a triple from $\text{SO}(2n+1)$ to $\text{Spin}(2n+1)$, related by multiplication with elements of C^3 , but generically these define the same holonomy modulo conjugation. This means that the corresponding states all have e trivial. However, if the four k_a associated with the points that do not belong to a certain line of the Fano plane are zero, then multiplication by the corresponding non-trivial element of C^3 changes the equivalence class of the triple. There will then be a further set of states with a corresponding non-trivial value of e [4].

It follows that for $m \in C^3$ trivial, all values of $e \in C^3$ are possible. A non-trivial value of e only appears when k_0 is odd, the four k_a associated with points not in the corresponding line are zero, and the three k_a associated with points in that line are even. For $m \in C^3$ non-trivial, e is either trivial or equal to m . The latter case appears only when k_0 is even, the four k_a associated with points not in the corresponding line are zero, and the remaining three k_a are odd. These pairs $(m, e) \in C^3 \times C^3$ are precisely those allowed by (2.11), as displayed in table 1, where 0 denotes the trivial element of \mathbb{Z}_2^3 and c is one of the seven non-trivial elements. We have also indicated the relevant finite quotient Γ of the stability group $\Omega_{m,e}$. It acts on the holonomies by permuting the k_a . (For $(m, e) = (0, c)$ one might think that the relevant group is $\text{SL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2$, but the second factor acts trivially on the holonomies.)

It is advantageous to consider all values of the rank of the gauge group, n , simultaneously: The number of distinguished markings of the $\mathfrak{so}(k)$ Dynkin diagram, i.e. $\dim V_{\mathfrak{so}(k)}$, equals the number of partitions of k into distinct odd parts [7], and this is most easily

(m, e)	Γ
$(0, 0)$	$\text{SL}_3(\mathbb{Z}_2)$
$(0, c)$	$\text{SL}_2(\mathbb{Z}_2)$
$(c, 0)$	$\text{SL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2$
(c, c)	$\text{SL}_2(\mathbb{Z}_2)$,

Table 1: Possible 't Hooft fluxes and the corresponding finite groups.

described by the generating function

$$P(q) = \sum_{k=1}^{\infty} q^k \dim V_{\text{so}(k)} = \prod_{k=1}^{\infty} (1 + q^{2k-1}), \quad (3.3)$$

which we decompose into its even and odd powers $P_{\text{even}} = \frac{1}{2}(P(q) + P(-q))$ and $P_{\text{odd}} = \frac{1}{2}(P(q) - P(-q))$, respectively. So rather than directly determining the multiplicities of bound states transforming in the unitary representation R of Γ , denoted by $M_{m,e}^R$ in this section, we will work with the generating functions

$$M_{m,e}^R(q) = \sum_{n=1}^{\infty} q^{2n+1} M_{m,e}^R. \quad (3.4)$$

These functions can be computed by a formula analogous to (2.14):

$$M_{m,e}^R(q) = \frac{1}{d} \sum_{[\gamma]} d_{[\gamma]} T_{m,e}^{[\gamma]}(q) \text{Tr}_R(\gamma), \quad (3.5)$$

where

$$T_{m,e}^{[\gamma]}(q) = \sum_{n=0}^{\infty} q^{2n+1} \text{Tr}_V(\gamma). \quad (3.6)$$

In the latter formula, the trace is over the space V of normalizable zero-energy states in the (m, e) -sector of the $\text{Spin}(2n + 1)$ theory.

We will now carry out these computations for all possible values of m and e separately.

3.1 The $(m, e) = (0, 0)$ states

The group $\Gamma = \text{SL}_3(\mathbb{Z}_2)$ is of order $d = 168$. The values of $\text{Tr}_R(\gamma)$ are given in the character table 2, where $c = \frac{1}{2}(-1 - i\sqrt{7})$ and \bar{c} is its complex conjugate. We have denoted the representations by their dimensionality in bold face, and the conjugacy classes by their cycle structure when acting on the seven k_a . From the above considerations follows that

$$\begin{aligned} T_{0,0}^{\mathbf{1}^7}(q) &= P_{\text{odd}}(q)P_{\text{even}}^7(q) + P_{\text{even}}(q)P_{\text{odd}}^7(q) \\ T_{0,0}^{\mathbf{1}^3\mathbf{2}^2}(q) &= P_{\text{odd}}(q)P_{\text{even}}^3(q)P_{\text{even}}^2(q^2) + P_{\text{even}}(q)P_{\text{odd}}^3(q)P_{\text{odd}}^2(q^2) \\ T_{0,0}^{\mathbf{1}^2\mathbf{4}}(q) &= P_{\text{odd}}(q)P_{\text{even}}(q)P_{\text{even}}(q^2)P_{\text{even}}(q^4) + P_{\text{even}}(q)P_{\text{odd}}(q)P_{\text{odd}}(q^2)P_{\text{odd}}(q^4) \\ T_{0,0}^{\mathbf{1}^3\mathbf{2}}(q) &= P_{\text{odd}}(q)P_{\text{even}}(q)P_{\text{even}}^2(q^3) + P_{\text{even}}(q)P_{\text{odd}}(q)P_{\text{odd}}^2(q^3) \\ T_{0,0}^{\mathbf{7}}(q) &= P_{\text{odd}}(q)P_{\text{even}}(q^7) + P_{\text{even}}(q)P_{\text{odd}}(q^7) \\ T_{0,0}^{\mathbf{7}'}(q) &= P_{\text{odd}}(q)P_{\text{even}}(q^7) + P_{\text{even}}(q)P_{\text{odd}}(q^7). \end{aligned} \quad (3.7)$$

conj.class	1^7	$1^3 2^2$	$1 2 4$	$1 3^2$	7	$7'$
cardinality	1	21	42	56	24	24
1	1	1	1	1	1	1
6	6	2	0	0	-1	-1
7	7	-1	-1	1	0	0
8	8	0	0	-1	1	1
3	3	-1	1	0	\bar{c}	c
$\bar{3}$	3	-1	1	0	c	\bar{c}

Table 2: Character table for $SL_3(\mathbb{Z}_2)$.

conj.class	1^3	$1 2$	3
cardinality	1	3	2
1	1	1	1
1'	1	-1	1
2	2	0	-1

Table 3: Character table for $SL_2(\mathbb{Z}_2)$.

The first (second) term in each expression corresponds to k_0 being odd (even) and the k_a being even (odd). The generating functions $M_{0,0}^R(q)$ are obtained through (3.5). As an example we give the resulting expression for the six-dimensional representation (here we have also used the second identity in (5.5)):

$$M_{0,0}^6 = \frac{1}{3584}P(q)^8 + \frac{1}{128}P(q)^4P(q^2)^2 + \frac{3}{256}P(q)^4P(-q^2)^2 + \frac{1}{32}P(q)^2P(-q^2)P(-q^4) + \frac{1}{7}P(q)P(q^7) - (q \leftrightarrow -q). \quad (3.8)$$

Note that $\sum_R \dim RM_{0,0}^R(q)$ reproduces the result in [4] as required for consistency.

3.2 The $(m, e) = (0, c)$ states

The group $\Gamma = SL_2(\mathbb{Z}_2) \simeq S_3$ (the symmetric group on three elements) is of order $d = 6$. In the character table 3, we have denoted the conjugacy classes by their cycle structure on the three non-zero k_a . The k_a are necessarily even, and k_0 is odd. It follows that

$$\begin{aligned} T_{0,c}^{1^3}(q) &= P_{\text{odd}}(q)P_{\text{even}}^3(q) \\ T_{0,c}^{1^2}(q) &= P_{\text{odd}}(q)P_{\text{even}}(q)P_{\text{even}}(q^2) \\ T_{0,c}^{\mathbf{3}}(q) &= P_{\text{odd}}(q)P_{\text{even}}(q^3). \end{aligned} \quad (3.9)$$

The generating functions $M_{0,c}^R(q)$ follow from (3.5).

3.3 The $(m, e) = (c, 0)$ states

The group $\Gamma = SL_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2 \simeq S_4$ (the symmetric group on four elements) is of order $d = 24$ with character table 4. The two entries in the notation for the conjugacy classes

conj.class	$(\mathbf{1}^3, \mathbf{1}^4)$	$(\mathbf{1}^3, \mathbf{2}^2)$	$(\mathbf{1} \mathbf{2}, \mathbf{1}^2 \mathbf{2})$	$(\mathbf{1} \mathbf{2}, \mathbf{4})$	$(\mathbf{3}, \mathbf{1} \mathbf{3})$
cardinality	1	3	6	6	8
$\mathbf{1}$	1	1	1	1	1
$\mathbf{1}'$	1	1	-1	-1	1
$\mathbf{2}$	2	2	0	0	-1
$\mathbf{3}$	3	-1	1	-1	0
$\mathbf{3}'$	3	-1	-1	1	0

Table 4: Character table for $\text{SL}_2(\mathbb{Z}_2)\langle\mathbb{Z}_2^2\rangle$.

refer to to the cycle structures on the three k_a in the line on the Fano plane and the four remaining k_a respectively. We get

$$\begin{aligned}
 T_{c,0}^{(\mathbf{1}^3, \mathbf{1}^4)}(q) &= P_{\text{odd}}^3(q)P_{\text{even}}^5(q) + P_{\text{even}}^3(q)P_{\text{odd}}^5(q) \\
 T_{c,0}^{(\mathbf{1}^3, \mathbf{2}^2)}(q) &= P_{\text{odd}}^3(q)P_{\text{even}}(q)P_{\text{even}}^2(q^2) + P_{\text{even}}^3(q)P_{\text{odd}}(q)P_{\text{odd}}^2(q^2) \\
 T_{c,0}^{(\mathbf{1} \mathbf{2}, \mathbf{1}^2 \mathbf{2})}(q) &= P_{\text{odd}}(q)P_{\text{odd}}(q^2)P_{\text{even}}^3(q)P_{\text{even}}(q^2) + P_{\text{even}}(q)P_{\text{even}}(q^2)P_{\text{odd}}^3(q)P_{\text{odd}}(q^2) \\
 T_{c,0}^{(\mathbf{1} \mathbf{2}, \mathbf{4})}(q) &= P_{\text{odd}}(q)P_{\text{odd}}(q^2)P_{\text{even}}(q)P_{\text{even}}(q^4) + P_{\text{even}}(q)P_{\text{even}}(q^2)P_{\text{odd}}(q)P_{\text{odd}}(q^4) \\
 T_{c,0}^{(\mathbf{3}, \mathbf{1} \mathbf{3})}(q) &= P_{\text{odd}}(q^3)P_{\text{even}}^2(q)P_{\text{even}}(q^3) + P_{\text{even}}(q^3)P_{\text{odd}}^2(q)P_{\text{odd}}(q^3), \tag{3.10}
 \end{aligned}$$

where the first (second) term in each expression corresponds to the three k_a in the line on the Fano plane being odd (even) and the four remaining k_a together with k_0 being even (odd). The generating functions $M_{c,0}^R(q)$ follow from (3.5).

3.4 The $(m, e) = (c, c)$ states

The group $\Gamma = \text{SL}_2(\mathbb{Z}_2) \simeq S_3$ is the same as in the $(m, e) = (0, c)$ case, but now the three non-zero k_a are odd, and k_0 is even. It follows that

$$\begin{aligned}
 T_{c,c}^{\mathbf{1}^3}(q) &= P_{\text{even}}(q)P_{\text{odd}}^3(q) \\
 T_{c,c}^{\mathbf{1}^2 \mathbf{2}}(q) &= P_{\text{even}}(q)P_{\text{odd}}(q)P_{\text{odd}}(q^2) \\
 T_{c,c}^{\mathbf{3}}(q) &= P_{\text{even}}(q)P_{\text{odd}}(q^3). \tag{3.11}
 \end{aligned}$$

The generating functions $M_{c,c}^R(q)$ follow from (3.5).

4. The $G = \text{Sp}(2n)$ theories

In this section we perform an analysis of the $\text{Sp}(2n)$ theories, similar to the one carried out for the $\text{Spin}(2n+1)$ theories in the previous section (see [4] for relevant background material). It turns out that it is convenient to treat all e values together; we therefore split the analysis into two cases, $m = 0$ and $m \neq 0$.

4.1 The $m = 0$ states

The bound states with $m = 0$ arise from semi-simple centralizers of the form [1, 4]

$$z \simeq \text{sp}(2k_1) \oplus \text{sp}(2k_2) \oplus \cdots \oplus \text{sp}(2k_8), \tag{4.1}$$

conj.class	1^8	$1^4 2^2$	2^4	$2^{4'}$	$1^2 3^2$	$1^2 24$	26	17	$17'$	4^2	$4^{2'}$
cardinality	1	42	42	7	224	168	224	192	192	168	84
1	1	1	1	1	1	1	1	1	1	1	1
3	3	-1	-1	3	0	1	0	c	\bar{c}	1	-1
$\bar{3}$	3	-1	-1	3	0	1	0	\bar{c}	c	1	-1
6	6	2	2	6	0	0	0	-1	-1	0	2
7	7	-1	-1	7	1	-1	1	0	0	-1	-1
8	8	0	0	8	-1	0	-1	1	1	0	0
1 · 7	7	3	-1	-1	1	1	-1	0	0	-1	-1
1' · 7	7	-1	3	-1	1	-1	-1	0	0	1	-1
2 · 7	14	2	2	-2	-1	0	1	0	0	0	-2
3 · 7	21	1	-3	-3	0	-1	0	0	0	1	1
3' · 7	21	-3	1	-3	0	1	0	0	0	-1	1

Table 5: Character table for $SL_3(\mathbb{Z}_2) \langle \mathbb{Z}_2^3 \rangle$.

where $\sum_i k_i = n$. One may think of k_1, \dots, k_8 as being associated with the eight corners of a cube, or equivalently \mathbb{Z}_2^3 (see [4] for more details). The corresponding commuting triples may be represented by the diagonal matrices [4]

$$\begin{aligned}
 U_1 &= \text{diag}(\mathbb{1}_{2k_1}, -\mathbb{1}_{2k_2}, \mathbb{1}_{2k_3}, -\mathbb{1}_{2k_4}, \mathbb{1}_{2k_5}, -\mathbb{1}_{2k_6}, \mathbb{1}_{2k_7}, -\mathbb{1}_{2k_8}), \\
 U_2 &= \text{diag}(\mathbb{1}_{2k_1}, \mathbb{1}_{2k_2}, -\mathbb{1}_{2k_3}, -\mathbb{1}_{2k_4}, \mathbb{1}_{2k_5}, \mathbb{1}_{2k_6}, -\mathbb{1}_{2k_7}, -\mathbb{1}_{2k_8}), \\
 U_3 &= \text{diag}(\mathbb{1}_{2k_1}, \mathbb{1}_{2k_2}, \mathbb{1}_{2k_3}, \mathbb{1}_{2k_4}, -\mathbb{1}_{2k_5}, -\mathbb{1}_{2k_6}, -\mathbb{1}_{2k_7}, -\mathbb{1}_{2k_8}).
 \end{aligned}
 \tag{4.2}$$

Since $U_i^2 = 1$ the action of $SL(3, \mathbb{Z})$ is reduced to $SL(3, \mathbb{Z}_2)$. The large gauge transformations act as reflections in the cube language and induce the natural $SL(3, \mathbb{Z}_2)$ action on $e = (e_1, e_2, e_3)$. These facts imply that the natural group to use to classify the $m = 0$ states is $SL(3, \mathbb{Z}_2) \times \mathbb{Z}_2^3$. This group has order 1344 and its character table is found in table 5,¹ where $c = \frac{1}{2}(-1 - i\sqrt{7})$ and \bar{c} is its complex conjugate. The non-trivial large gauge transformations constitute the conjugacy class $2^{4'}$ of cardinality 7. We see that these act trivially, i.e. imply $e = 0$, on the first six representations, which therefore can be identified with representations of the $SL_3(\mathbb{Z}_2)$ stability group. The last five representations contain the states where e takes one of the 7 non-zero values. We denote these representations as $\mathbf{R} \cdot 7$, where \mathbf{R} is a representation of the $SL_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2 \simeq S_4$ stability group.

Just as for $Spin(2n+1)$, it is convenient to consider all values of n simultaneously. Indeed, the number of distinguished markings of the $sp(2k)$ Dynkin diagram, i.e. $\dim V_{sp(2k)}$, equals the number of partitions of $2k$ into distinct even parts. This is most easily described by the generating function

$$Q(q) = \sum_{k=1}^{\infty} q^{2k} \dim V_{sp(2k)} = \prod_{n=1}^{\infty} (1 + q^{2n}).
 \tag{4.3}$$

¹This table is derived in [12], or can be obtained using the GAP computational algebra system [13].

We can then associate a generating function to each conjugacy class:

$$\begin{aligned}
 T_{m=0}^{\mathbf{1}^8} &= q Q(q)^8, \\
 T_{m=0}^{\mathbf{1}^4\mathbf{2}^2} &= q Q(q)^4 Q(q^2)^2, \\
 T_{m=0}^{\mathbf{2}^4} &= q Q(q^2)^4, \\
 T_{m=0}^{\mathbf{2}^{4'}} &= q Q(q^2)^4, \\
 T_{m=0}^{\mathbf{1}^2\mathbf{3}^2} &= q Q(q)^2 Q(q^3)^2, \\
 T_{m=0}^{\mathbf{1}^2\mathbf{2}^4} &= q Q(q)^2 Q(q^2) Q(q^4), \\
 T_{m=0}^{\mathbf{2}^6} &= q Q(q^2) Q(q^6), \\
 T_{m=0}^{\mathbf{1}^7} &= q Q(q) Q(q^7), \\
 T_{m=0}^{\mathbf{1}^{7'}} &= q Q(q) Q(q^7), \\
 T_{m=0}^{\mathbf{4}^2} &= q Q(q^4)^2, \\
 T_{m=0}^{\mathbf{4}^{2'}} &= q Q(q^4)^2,
 \end{aligned} \tag{4.4}$$

where the prefactor q is introduced to facilitate the comparison with the Spin(2n+1) results of the previous section. The generating functions for the number of states transforming in the various unitary representations can be computed by a formula analogous to (3.5):

$$N_{m=0}^R(q) = \frac{1}{d} \sum_{[\gamma]} d_{[\gamma]} T_{m=0}^{[\gamma]}(q) \text{Tr}_R(\gamma). \tag{4.5}$$

As an example, we write explicitly the generating function for the $(e, m) = (0, 0)$ states transforming in the $\mathbf{6}$ of $\text{SL}_3(\mathbb{Z}_2)$:

$$N_{0,0}^{\mathbf{6}}(q) = \frac{q}{224} (Q(q)^8 + 14Q(q)^4 Q(q^2)^2 + 21Q(q^2)^4 + 28Q(q^4)^2 - 64Q(q)Q(q^7)). \tag{4.6}$$

4.2 The $m \neq 0$ states

The bound states with $m \neq 0$ arise from semi-simple centralizers of the form [11, 4]

$$Z \simeq \text{so}(k_1) \oplus \text{so}(k'_1) \oplus \text{sp}(2k_2) \oplus \text{sp}(2k'_2) \oplus \dots \oplus \text{sp}(2k_4) \oplus \text{sp}(2k'_4). \tag{4.7}$$

One may think of the k_i and the k'_i as being associated with the points of two parallel planes (determined by m) on a cube. The corresponding holonomies can be found in [4, 11]. They are at most of order 2, so just as for $m = 0$, only the mod 2 reduction of the $\text{SL}_3(\mathbb{Z})$ mapping class group is relevant. Furthermore the \mathbb{Z}_2^2 factor of the $\text{SL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2^2$ stability subgroup of m acts trivially on the holonomies, so we need only consider the group $\text{SL}_2(\mathbb{Z}_2)$. Including also the non-trivial large gauge transformations, we are led to consider the finite group

$$\Gamma = \text{SL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2, \tag{4.8}$$

of order $d = 12$. In the character table 6, the first (second) entry in the notation for the conjugacy classes indicates the cycle structure on k_1 and k'_1 (k_2, k_3, k_4 and k'_2, k'_3, k'_4). The non-trivial large gauge transformation constitutes the $(\mathbf{2}, \mathbf{2}^3)$ conjugacy class of cardinality 1. It is trivially represented on the first three representations, which thus have $e = 0$, and

conj.class	$(1^2, 1^6)$	$(1^2, 1^2 2^2)$	$(1^2, 3^2)$	$(2, 2^3)$	$(2, 2^3)'$	$(2, 6)$
cardinality	1	3	2	1	3	2
$\mathbf{1}_0$	1	1	1	1	1	1
$\mathbf{1}'_0$	1	-1	1	1	-1	1
$\mathbf{2}_0$	2	0	-1	2	0	-1
$\mathbf{1}_m$	1	1	1	-1	-1	-1
$\mathbf{1}'_m$	1	-1	1	-1	1	-1
$\mathbf{2}_m$	2	0	-1	-2	0	1

Table 6: Character table for $SL_2(\mathbb{Z}_2) \times \mathbb{Z}_2$.

non-trivially represented on the last three representations, which thus have $e = m$. The generating functions associated with the conjugacy classes are

$$\begin{aligned}
 T_{m \neq 0}^{(1^2, 1^6)} &= qP(q^2)^2 Q(q^2)^6, \\
 T_{m \neq 0}^{(1^2, 1^2 2^2)} &= qP(q^2)^2 Q(q^2)^2 Q(q^4)^2, \\
 T_{m \neq 0}^{(1^2, 3^2)} &= qP(q^2)^2 Q(q^6)^2, \\
 T_{m \neq 0}^{(2, 2^3)} &= qP(q^4) Q(q^4)^3, \\
 T_{m \neq 0}^{(2, 2^3)'} &= qP(q^4) Q(q^4)^3, \\
 T_{m \neq 0}^{(2, 6)} &= qP(q^4) Q(q^{12}).
 \end{aligned}
 \tag{4.9}$$

The generating functions for the unitary representations are obtained as in (4.5).

5. S-duality

The requirements of S -duality are easy to verify by expanding out the expressions obtained in the previous sections for $M_{m,e}^R(q)$ and $N_{m,e}^R(q)$ as power series in q . One finds that

$$\begin{aligned}
 M_{0,0}^1 &= N_{0,0}^1 = q + q^3 + 2q^5 + 4q^7 + 8q^9 + 13q^{11} + 26q^{13} + 44q^{15} + 80q^{17} + \dots \\
 M_{0,0}^3 &= N_{0,0}^3 = 2q^{15} + 7q^{17} + \dots \\
 M_{0,0}^{\bar{3}} &= N_{0,0}^{\bar{3}} = 2q^{15} + 7q^{17} + \dots \\
 M_{0,0}^6 &= N_{0,0}^6 = q^5 + 2q^7 + 7q^9 + 14q^{11} + 34q^{13} + 68q^{15} + 146q^{17} + \dots \\
 M_{0,0}^7 &= N_{0,0}^7 = q^{11} + 6q^{13} + 16q^{15} + 43q^{17} + \dots \\
 M_{0,0}^8 &= N_{0,0}^8 = q^9 + 4q^{11} + 12q^{13} + 32q^{15} + 80q^{17} + \dots \\
 M_{0,c}^1 &= N_{c,0}^1 = q + q^3 + 2q^5 + 3q^7 + 7q^9 + 10q^{11} + 19q^{13} + 29q^{15} + 50q^{17} + \dots \\
 M_{0,c}^{1'} &= N_{c,0}^{1'} = q^{11} + 3q^{13} + 7q^{15} + 13q^{17} + \dots \\
 M_{0,c}^2 &= N_{c,0}^2 = q^5 + 2q^7 + 5q^9 + 9q^{11} + 18q^{13} + 31q^{15} + 57q^{17} + \dots \\
 M_{c,0}^1 &= N_{0,c}^1 = q^3 + 2q^5 + 6q^7 + 12q^9 + 27q^{11} + 55q^{13} + 112q^{15} + 215q^{17} + \dots \\
 M_{c,0}^{1'} &= N_{0,c}^{1'} = q^9 + q^{11} + 7q^{13} + 16q^{15} + 47q^{17} + \dots \\
 M_{c,0}^2 &= N_{0,c}^2 = q^5 + 2q^7 + 7q^9 + 18q^{11} + 45q^{13} + 100q^{15} + 222q^{17} + \dots \\
 M_{c,0}^3 &= N_{0,c}^3 = 2q^7 + 6q^9 + 19q^{11} + 46q^{13} + 116q^{15} + 257q^{17} + \dots
 \end{aligned}$$

$$\begin{aligned}
M_{c,0}^{\mathbf{3}'} &= N_{0,c}^{\mathbf{3}'} = q^9 + 5q^{11} + 18q^{13} + 52q^{15} + 137q^{17} + \dots \\
M_{c,c}^{\mathbf{1}} &= N_{c,c}^{\mathbf{1}} = q^3 + q^5 + 3q^7 + 5q^9 + 10q^{11} + 16q^{13} + 29q^{15} + 45q^{17} + \dots \\
M_{c,c}^{\mathbf{1}'} &= N_{c,c}^{\mathbf{1}'} = q^9 + q^{11} + 4q^{13} + 7q^{15} + 15q^{17} + \dots \\
M_{c,c}^{\mathbf{2}} &= N_{c,c}^{\mathbf{2}} = q^5 + 2q^7 + 4q^9 + 9q^{11} + 17q^{13} + 31q^{15} + 55q^{17} + \dots,
\end{aligned}$$

where c is an arbitrary non-trivial element of \mathbb{Z}_2^3 . This means e.g. that for each non-trivial $c \in \mathbb{Z}_2^3$, the $m = 0, e = c$ states of the Spin(17) theory comprise 57 doublets of the $SL_2(\mathbb{Z}_2)$ stability subgroup of the mapping class group.

It is actually possible to prove these identities to all orders in q ; the analysis reveals a rich structure of combinatorial infinite product identities dating back to Euler, Ramanujan and others. S -duality between the $m = 0$ states in the $Sp(2n)$ theory and the $e = 0$ states in the Spin($2n + 1$) theory is equivalent to linear combinations of the following identities:

$$\begin{aligned}
16qQ(q)^8 &= P(q)^8 - (q \leftrightarrow -q), \\
8qQ(q)^4Q(q^2)^2 &= P(q)^4P(q^2)^2 - (q \leftrightarrow -q), \\
4qQ(q)^2Q(q^3)^2 &= P(q)^2P(q^3)^2 - (q \leftrightarrow -q), \\
4qQ(q)^2Q(q^2)Q(q^4) &= P(q)^2P(q^2)P(q^4) - (q \leftrightarrow -q), \\
2qQ(q)Q(q^7) &= P(q)P(q^7) - (q \leftrightarrow -q),
\end{aligned} \tag{5.1}$$

as well as

$$\begin{aligned}
8qQ(q^2)^4 &= P(q)^4P(-q^2)^2 - (q \leftrightarrow -q), \\
4qQ(q^2)Q(q^6) &= P(q)^2P(-q^3)^2 - (q \leftrightarrow -q), \\
4qQ(q^4)^2 &= P(q)^2P(-q^2)P(-q^4) - (q \leftrightarrow -q).
\end{aligned} \tag{5.2}$$

Before we proceed let us make a few remarks about these identities. The first identity in (5.1) is Jacobi’s famous *aequatio identica satis abstrusa*, which also appeared in our previous paper [4]. The other identities are “ $SL_3(\mathbb{Z}_2)$ refinements” of this identity. We note that identities of the type (5.1) have recently enjoyed a renewed interest in the mathematics literature. In particular, in the work of Farkas and Kra [14] identity 1, 3 and 5 of (5.1) above were referred to as ‘a curious property of’ ‘eight’, ‘three’, and ‘seven’, respectively. In their work, these three identities were treated in a case-by-case manner. The fact that we have found a connection via $SL_3(\mathbb{Z}_2)$ and S -duality between these and other identities might be of some interest.

For S -duality between the states with $m \neq 0$ in the $Sp(2n)$ theory and the $e \neq 0$ states in the Spin($2n + 1$) theory one similarly needs the identities

$$\begin{aligned}
8qP(q^2)^2Q(q^2)^6 &= P(q)^4 - (q \leftrightarrow -q), \\
4qP(q^2)^2Q(q^2)^2Q(q^4)^2 &= P(q)^2P(q^2) - (q \leftrightarrow -q), \\
2qP(q^2)^2Q(q^6)^2 &= P(q)P(q^3) - (q \leftrightarrow -q),
\end{aligned} \tag{5.3}$$

as well as

$$\begin{aligned}
4qP(q^4)Q(q^4)^3 &= P(q)^2P(-q^2) - (q \leftrightarrow -q), \\
2qP(q^4)Q(q^{12}) &= P(q)P(-q^3) - (q \leftrightarrow -q).
\end{aligned} \tag{5.4}$$

Note that the above identities can be rewritten using the elementary relations

$$\begin{aligned} Q(q^2)P(q^2) &= Q(q), \\ P(-q)P(q) &= P(-q^2), \end{aligned} \tag{5.5}$$

together with Euler's identity

$$Q(q)P(-q^2) = 1. \tag{5.6}$$

All of the above identities are special cases of certain identities among theta functions (they can also be proved starting from entries 29 and 30 in section 16 of [15]). The following identity

$$\begin{aligned} 2q h(a, b) \prod_{n=1}^{\infty} (a+q^{2n}) \left(\frac{1}{a}+q^{2n}\right) (b+q^{2n}) \left(\frac{1}{b}+q^{2n}\right) (ab+q^{2n}) \left(\frac{1}{ab}+q^{2n}\right) (1+q^{2n})^2 \\ = \prod_{n=1}^{\infty} (a+q^{2n-1}) \left(\frac{1}{a}+q^{2n-1}\right) (b+q^{2n-1}) \left(\frac{1}{b}+q^{2n-1}\right) (ab+q^{2n-1}) \left(\frac{1}{ab}+q^{2n-1}\right) (1+q^{2n-1})^2 \\ - \prod_{n=1}^{\infty} (a-q^{2n-1}) \left(\frac{1}{a}-q^{2n-1}\right) (b-q^{2n-1}) \left(\frac{1}{b}-q^{2n-1}\right) (ab-q^{2n-1}) \left(\frac{1}{ab}-q^{2n-1}\right) (1-q^{2n-1})^2, \end{aligned} \tag{5.7}$$

where $h(a, b) = (1+a)(1+b)(1+\frac{1}{ab})$, follows from the theta function result

$$\begin{aligned} \theta_2(\alpha, q) \theta_2(\beta, q) \theta_2(\alpha + \beta, q) \theta_2(0, q) = \\ \theta_3(\alpha, q) \theta_3(\beta, q) \theta_3(\alpha + \beta, q) \theta_3(0, q) - \theta_4(\alpha, q) \theta_4(\beta, q) \theta_4(\alpha + \beta, q) \theta_4(0, q). \end{aligned} \tag{5.8}$$

When is the l.h.s. of (5.7) of the form $\prod_{n=1}^{\infty} \prod_i (1 + q^{2n\ell_i})$ for integers ℓ_i with $\sum_i \ell_i = 8$? It turns out that there are only five solutions to this requirement, namely

$$(a, b) \in \{(1, 1), (i, 1), (-e^{i\pi/3}, 1), (-e^{i\pi/4}, i), (-e^{i\pi/7}, -e^{9i\pi/7})\}. \tag{5.9}$$

The resulting identities precisely correspond to the above expressions (5.1). It is interesting to note that essentially the same identity (5.7) was also needed to show the equality between the number of vacuum states in the $\mathcal{N} = 1^*$ mass-deformed $\mathcal{N} = 4$ Sp(2n) and SO(2n) theories, as required by S -duality [16].

The identity

$$2\theta_2(\alpha + \beta, q^2)\theta_2(\alpha - \beta, q^2) = \theta_3(\alpha, q)\theta_3(\beta, q) - \theta_4(\alpha, q)\theta_4(\beta, q), \tag{5.10}$$

with $\alpha = \beta$ can be rewritten as

$$\begin{aligned} 4q \frac{(1+a^2)}{a} \prod_{n=1}^{\infty} (a^2 + q^{4n}) \left(\frac{1}{a^2} + q^{4n}\right) (1 + q^{4n})^2 \\ = \prod_{n=1}^{\infty} \left(\frac{1}{a} + q^{2n-1}\right)^2 (a + q^{2n-1})^2 (1 - q^{4n-2})^2 - (q \leftrightarrow -q). \end{aligned} \tag{5.11}$$

For the two parameter choices

$$a \in \left\{1, e^{i\pi/3}\right\}, \tag{5.12}$$

the resulting identities are identical to the first and second entries in (5.2).

From the identity

$$2\theta_2(2\alpha, q^4) = \theta_3(\alpha, q) - \theta_4(\alpha, q), \tag{5.13}$$

with $\alpha = 0$ one easily obtains the final identity in (5.2).

Next we consider the following identity obtained from (5.10)

$$\begin{aligned} 2q g(a, b) \prod_{n=1}^{\infty} (ab + q^{4n}) \left(\frac{1}{ab} + q^{4n}\right) \left(\frac{b}{a} + q^{4n}\right) \left(\frac{a}{b} + q^{4n}\right) (1 + q^{4n})^2 (1 + q^{4n-2}) \\ = \prod_{n=1}^{\infty} (a + q^{2n-1}) \left(\frac{1}{a} + q^{2n-1}\right) (b + q^{2n-1}) \left(\frac{1}{b} + q^{2n-1}\right) - (q \leftrightarrow -q) \end{aligned} \tag{5.14}$$

where $g(a, b) = (1 + \frac{1}{ab})(1 + \frac{b}{a})$. There are only three choices of the parameters for which we get integer exponents as above, namely

$$(a, b) \in \{(1, 1), (1, i), (1, -e^{i\pi/3})\}. \tag{5.15}$$

The resulting identities precisely correspond to the identities in (5.3).

From the identity (5.13) one deduces

$$\begin{aligned} 2q \frac{(1 + a^2)}{a} \prod_{n=1}^{\infty} (1 + q^{8n-4})(a^2 + q^{8n}) \left(\frac{1}{a^2} + q^{8n}\right) (1 + q^{8n}) \\ = \prod_{n=1}^{\infty} (1 - q^{4n-2}) \left(\frac{1}{a} + q^{2n-1}\right) (a + q^{2n-1}) - (q \leftrightarrow -q) \end{aligned} \tag{5.16}$$

There are two parameter choices that we need, namely

$$a \in \{-1, e^{i\pi/3}\}. \tag{5.17}$$

The resulting expressions reproduce the identities in (5.4). Note that the theta function identities used here, (5.8), (5.10) and (5.13), are parameter deformations of the identities (5.51-53) used in [4].

Acknowledgments

M.H. is supported by grants from the Göran Gustafsson foundation and the Swedish Research Council.

N.W. is supported by a grant from the Swedish Research Council.

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